TITLE: Zeros of the Hypergeometric Polynomial F[-n, b; c; z]

Zeros of the hypergeometric polynomial $F(-n, b; c; z)$

K. Driver* **and K. Jordaan**

School of Mathematics, University of the Witwatersrand, Johannesburg, South Africa.
036kad@cosmos.wits.ac.za, 036jord@cosmos.wits.ac.za

Abstract

Our interest lies in describing the zero behaviour of Gauss hypergeometric polynomials $F(-n, b; c; z)$ where b and c are arbitrary parameters. In general, this problem has not been solved and even when b and c are both real, the only cases that have been fully analysed impose additional restrictions on b and c. We review recent results that have been proved for the zeros of several classes of hypergeometric polynomials $F(-n, b; c; z)$ where b and c are real. We show that the number of real zeros of $F(-n, b; c; z)$ for arbitrary real values of the parameters b and c, as well as the intervals in which these zeros (if any) lie, can be deduced from corresponding results for Jacobi polynomials.

1 Introduction

The Gauss hypergeometric function, or ${}_2F_1$, is defined by

$$F(a, b; c; z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1,$$

where a, b and c are complex parameters and

$$(\alpha)_k = \alpha(\alpha+1)\dots(\alpha+k-1) = \Gamma(\alpha+k)/\Gamma(\alpha)$$

is Pochhammer's symbol. When $a = -n$ is a negative integer, the series terminates and reduces to a polynomial of degree n, called a hypergeometric polynomial. Our focus lies in the location of the zeros $F(-n, b; c; z)$ for real values of b and c.

Hypergeometric polynomials are connected with several different types of orthogonal polynomials, notably Chebyshev, Legendre, Gegenbauer and Jacobi polynomials. In the cases of Chebyshev and Legendre polynomials, the connection demands fixed special values of the parameters b and c, namely, (cf. [1], p.561)

$$F\left(-n, n; \frac{1}{2}; z\right) = T_n(1-2z)$$

and

$$F(-n, n+1; 1; z) = P_n(1-2z),$$

*Research of the first author is supported by the John Knopfmacher Centre for Applicable Analysis and Number Theory, University of the Witwatersrand.

respectively. However, in the cases of Gegenbauer and Jacobi polynomials, we have

$$F\left(-n, n+2\lambda;\ \lambda+\frac{1}{2};\ z\right) = \frac{n!}{(2\lambda)_n} C_n^{\lambda}(1-2z) \tag{1.1}$$

and

$$F\left(-n, \alpha+\beta+1+n;\ \alpha+1;\ z\right) = \frac{n!}{(\alpha+1)_n} \mathcal{P}_n^{(\alpha,\beta)}(1-2z), \tag{1.2}$$

respectively. Since the zeros of orthogonal polynomials are well understood, we expect the connections (1.1) and (1.2) to be very useful in analysing the zeros of $F(-n, b; c; z)$. Conversely, if the zeros of $F(-n, b; c; z)$ are known, this leads to new information about the zero distribution of Gegenbauer or Jacobi polynomials for values of their parameters that lie outside the range of orthogonality of these polynomials.

This paper is organized as follows. In Section 2 we give a self-contained review of recent results regarding the zeros of several special classes of hypergeometric polynomials. Section 3 contains results originally due to Klein [9] which detail the numbers and location of real zeros of $F(-n, b; c; z)$ for arbitrary real values of b and c. We provide simple proofs using results proved in [13].

2 Zeros of special classes of hypergeometric polynomials

We begin with a few general remarks. Since we shall assume throughout our discussion that b and c are real parameters, we know that all zeros of $F(-n, b; c; z)$ must occur in complex conjugate pairs. In particular, if n is odd, F must always have at least one real zero. Further, if $b = -m$ where $m < n$, $m \in \mathbb{N}$, $F(-n, b; c; z)$ reduces to a polynomial of degree m. However, since we are interested in the behaviour of the zeros of $F(-n, b; c; z)$ as b and/or c vary through real values, we shall adopt the convention that $F(-n, -m; c; z) = \lim_{b\to -m} F(-n, b; c; z)$. This ensures that the zeros of F vary continuously with b and c. Note also that $F(-n, b; c; z)$ is not defined when $c = 0, -1, \ldots, -n+1$. Regarding the multiplicity of zeros, a hypergeometric function $w = F(a, b; c; z)$ satisfies the differential equation

$$z(1-z)w'' + \big[c - (a+b+1)z\big]w' - abw = 0,$$

so if $w(z_0) = w'(z_0) = 0$ at some point $z_0 \neq 0$ or 1, it would follow that $w \equiv 0$. Thus multiple zeros of $F(-n, b; c; z)$ can only occur at $z = 0$ or 1.

2.1 Quadratic transformations

The class of hypergeometric polynomials that admit a quadratic transformation is specified by a necessary and sufficient condition due to Kummer (cf. [1], p.560). There are

twelve polynomials in this class (cf. [14], p.124)

$$
\begin{array}{lll}
F(-n,b;\ 2b;\ z) & F(-n,b;\ -n-b+1;\ z) & F\left(-n,b;\ \frac{-n+b+1}{2};\ z\right) \\
F\left(-n,b;\ \frac{1}{2};\ z\right) & F\left(-n,-n+\frac{1}{2};\ c;\ z\right) & F\left(-n,b;\ -n+b+\frac{1}{2};\ z\right) \\
F\left(-n,b;\ \frac{3}{2};\ z\right) & F\left(-n,-n-\frac{1}{2};\ c;\ z\right) & F\left(-n,b;\ -n+b-\frac{1}{2};\ z\right) \\
F(-n,b;\ -2n;\ z) & F(-n,b;\ b+n+1;\ z) & F(-n,n+1;\ c;\ z)\,.
\end{array}
$$

The most important polynomial in this class is $F(-n, b;\ 2b;\ z)$ because complete analysis of its zero distribution for all real values of b (cf. [4], [5]) leads to corresponding results for the zeros of the Gegenbauer polynomials $C_n^\lambda(z)$ for all real values of the parameter λ (cf. [6]).

Theorem 2.1. *Let $F = F(-n, b;\ 2b;\ z)$ where b is real.*

(i) *For $b > -\frac{1}{2}$, all zeros of $F(-n, b;\ 2b;\ z)$ are simple and lie on the circle $|z-1| = 1$.*

(ii) *For $-\frac{1}{2} - j < b < \frac{1}{2} - j$, $j = 1, 2, \ldots \left[\frac{n}{2}\right] - 1$, $(n-2j)$ zeros of F lie on the circle $|z-1| = 1$. If $j = 2k$ is even, there are k non-real zeros of F in each of the four regions bounded by the circle $|z-1| = 1$ and the real axis. If $j = 2k+1$ is odd, there are k non-real zeros of F in each of the four regions described above and the remaining two zeros are real.*

(iii) *If n is even, for $-\left[\frac{n}{2}\right] < b < -\left[\frac{n}{2}\right] + \frac{1}{2}$, no zeros of F lie on $|z-1| = 1$. If $n = 4k$, all zeros of F are non-real whereas if $n = 4k+2$, two zeros of F are real and $4k$ are non-real. If n is odd, for $-1 - \left[\frac{n}{2}\right] < b < -\left[\frac{n}{2}\right] + \frac{1}{2}$, only the fixed real zero of F at $z = 2$ lies on $|z-1| = 1$. If $n = 4k+1$, $n - 1 = 4k$ zeros of F are non-real whereas if $n = 4k+3$, two further zeros are real and the remaining $4k$ are non-real.*

(iv) *For $j - n < b < j - n + 1$, $j = 1, 2, \ldots \left[\frac{n}{2}\right] - 1$, $(n-2j)$ zeros of F are real and greater than 1. If $j = 2k$ is even, all remaining $2j$ zeros of F are non-real with k zeros in each of the regions described above; while if $j = 2k+1$, $4k$ zeros are non-real as before and 2 are real.*

(v) *For $b < 1 - n$, all zeros of $F(-n, b;\ 2b;\ z)$ are real and greater than 1. As $b \to -\infty$, all the zeros of F converge to the point $z = 2$.*

An analogous theorem which describes the behaviour of the zeros of $C_n^\lambda(z)$ can be found in [6], Section 3 or [7], Theorem 1.2.

For the polynomial $F\left(-n, b;\ \frac{1}{2};\ z\right)$ the following result has been proved in [7], Theorem 2.3.

Theorem 2.2. *Let $F = F\left(-n, b;\ \frac{1}{2};\ z\right)$ with b real.*

(i) *For $b > n - \frac{1}{2}$, all n zeros of F are real and simple and lie in $(0,1)$.*

(ii) *For $n - \frac{1}{2} - j < b < n + \frac{1}{2} - j$, $j = 1, 2, \ldots, n-1$, $(n-j)$ zeros of F lie in $(0,1)$ and the remaining j zeros of F form $\left[\frac{j}{2}\right]$ non-real complex pairs of zeros and one real zero lying in $(1, \infty)$ when j is odd.*

(iii) *For* $0 < b < \frac{1}{2}$, *F has* $\left[\frac{n}{2}\right]$ *non-real complex conjugate pairs of zeros with one real zero in* $(1,\infty)$ *when n is odd.*

(iv) *For* $-j < b < -j+1$, $j = 1,2,\ldots,n-1$, *F has exactly j real negative zeros. There is exactly one further real zero greater than 1 only when* $(n-j)$ *is odd and all the remaining zeros of F are non-real.*

(v) *For* $b < 1-n$, *all zeros of F are real and negative and converge to zero as* $b \to -\infty$.

A very similar theorem is proved for the zeros of $F\left(-n,b;\ \frac{3}{2};\ z\right)$ in [7], Theorem 2.4 with only minor differences of detail.

For the hypergeometric polynomial $F(-n,b;\ -2n;\ z)$, less complete results have been proved. We have (cf. [8] Theorem 3.1 and Corollary 3.2) the following.

Theorem 2.3. *Let* $F = F\left(-n,b;\ -2n;\ z\right)$ *with b real.*

(i) *For* $b > 0$, *F has n non-real zeros if n is even whereas if n is odd, F has exactly one real negative zero and the remaining* $(n-1)$ *zeros of F are all non-real.*

(ii) *For* $-n < b < 0$, *if* $-k < b < -k+1$, $k = 1,\ldots,n$, *F has k real zeros in the interval* $(1,\infty)$. *In addition, if* $(n-k)$ *is even, F has* $(n-k)$ *non-real zeros whereas if* $(n-k)$ *is odd, F has one real negative zero and* $(n-k-1)$ *non-real zeros.*

(iii) *For* $-n > b > -2n$, *if* $-n-k > b > -n-k-1$, $k = 0,1,\ldots,n-1$, *F has* $(n-k)$ *real zeros in the interval* $(1,\infty)$. *In addition, if k is even F has k non-real zeros while if k is odd, F has one real zero in* $(0,1)$ *and* $(k-1)$ *non-real zeros.*

(iv) *For* $b < -2n$, *all n zeros of F are non-real for n even whereas for n odd, F has exactly one real zero in the interval* $(0,1)$.

The identities (cf. [7], Lemma 2.1)

$$F(-n,b;\ c;\ 1-z) = \frac{(c-b)_n}{(c)_n} F(-n,b;\ 1-n+b-c;\ z) \tag{2.1}$$

and

$$F(-n,b;\ c;\ z) = \frac{(b)_n}{(c)_n}(-z)^n F\left(-n, 1-c-n;\ 1-b-n;\ \frac{1}{z}\right) \tag{2.2}$$

hold for b and c real, $c \neq \{0,-1,\ldots,-n+1\}$. Applying (2.1) and (2.2) to each of the polynomials $F(-n,b;\ 2b;\ z)$, $F\left(-n,b;\ \frac{1}{2};\ z\right)$, $F\left(-n,b;\ \frac{3}{2};\ z\right)$ and $F(-n,b;\ -2n;\ z)$ in turn, we obtain the remaining eight polynomials in the quadratic class. It is then an easy task to deduce analogous results for their zero distribution.

A similar set of results has been proved for the sixteen hypergeometric polynomials in the cubic class. Again, this class arises from a necessary and sufficient condition (cf. [2], p.67) and details can be found in [7].

3 The real zeros of $F(-n,b;\ c;\ z)$ for b and c real

The results proved below are due to Klein [9] who considered the zeros of more general hypergeometric functions (not necessarily polynomials). Klein's proof is geometric and

difficult to penetrate. A more transparent perspective in the polynomial case may be provided by the approach given here.

The classical equation linking the hypergeometric polynomial $F(-n, b; c; z)$ with Jacobi polynomials $\mathcal{P}_n^{(\alpha,\beta)}(z)$ is given by (1.2). We will find an alternative expression (cf. [12], p.464, eqn. (142))

$$F(-n, b; c; z) = \frac{n! z^n}{(c)_n} \mathcal{P}_n^{(\alpha,\beta)} \left(1 - \frac{2}{z}\right), \tag{3.1}$$

where $\alpha = -n-b$ and $\beta = b-c-n$, more suited to our analysis. The number of real zeros of $\mathcal{P}_n^{(\alpha,\beta)}(x)$ in the intervals $(-1, 1)$, $(-\infty, 1)$ and $(1, \infty)$ are given by the Hilbert-Klein formulas (cf. [13], p.145, Theorem 6.72), also known to Stieltjes. We use Klein's symbol

$$E(u) = \begin{cases} 0 & \text{if } u \leq 0 \\ [u] & \text{if } u > 0,\ u \neq \text{integer} \\ u-1 & \text{if } u = 1, 2, 3, \ldots \end{cases} .$$

Noting that under the linear fractional transformation $w = 1 - 2/z$, the intervals $1 < w < \infty$, $-\infty < w < -1$ and $-1 < w < 1$ correspond to $-\infty < z < 0$, $0 < z < 1$ and $1 < z < \infty$ respectively, we can use equation (3.1) to rephrase the Hilbert-Klein formulas for hypergeometric polynomials.

Theorem 3.1. *Let $b, c \in \mathbb{R}$ with $b, c, c-b \neq 0, -1, \ldots, -n+1$. Let*

$$X = E\left\{\frac{1}{2}(|1-c| - |n+b| - |b-c-n| + 1)\right\} \tag{3.2}$$

$$Y = E\left\{\frac{1}{2}(-|1-c| + |n+b| - |b-c-n| + 1)\right\} \tag{3.3}$$

$$Z = E\left\{\frac{1}{2}(-|1-c| - |n+b| + |b-c-n| + 1)\right\}. \tag{3.4}$$

Then the numbers of zeros of $F(-n, b; c; z)$ in the intervals $(1, \infty)$, $(0, 1)$ and $(-\infty, 0)$ respectively are

$$N_1 = \begin{cases} 2[(X+1)/2] & \text{if } (-1)^n \binom{-b}{n}\binom{b-c}{n} > 0 \\ 2[X/2] + 1 & \text{if } (-1)^n \binom{-b}{n}\binom{b-c}{n} < 0 \end{cases} \tag{3.5}$$

$$N_2 = \begin{cases} 2[(Y+1)/2] & \text{if } \binom{-c}{n}\binom{b-c}{n} > 0 \\ 2[Y/2] + 1 & \text{if } \binom{-c}{n}\binom{b-c}{n} < 0 \end{cases} \tag{3.6}$$

$$N_3 = \begin{cases} 2[(Z+1)/2] & \text{if } \binom{-c}{n}\binom{-b}{n} > 0 \\ 2[Z/2] + 1 & \text{if } \binom{-c}{n}\binom{-b}{n} < 0. \end{cases} \tag{3.7}$$

Proof: The expressions all follow immediately from the Hilbert-Klein formulas (cf. [13], p.145, Thm. 6.72) together with equation (3.1). □

Theorem 3.2. *Let $F = F(-n, b; c; z)$ where $b, c \in \mathbb{R}$ and $c > 0$.*

(i) For $b > c+n$, all zeros of F are real and lie in the interval $(0,1)$.

(ii) For $c < b < c+n$, $c+j-1 < b < c+j$, $j = 1,2,\ldots,n$; F has j real zeros in $(0,1)$. The remaining $(n-j)$ zeros of F are all non-real if $(n-j)$ is even while if $(n-j)$ is odd, F has $(n-j-1)$ non-real zeros and one additional real zero in $(1,\infty)$.

(iii) For $0 < b < c$, all the zeros of F are non-real if n is even, while if n is odd, F has one real zero in $(1,\infty)$ and the other $(n-1)$ zeros are non-real.

(iv) For $-n < b < 0$, $-j < b < -j+1$, $j = 1,2,\ldots,n$, F has j real negative zeros. The remaining $(n-j)$ zeros of F are all non-real if $(n-j)$ is even, while if $(n-j)$ is odd, F has $(n-j-1)$ non-real zeros and one additional real zero in $(1,\infty)$.

(v) For $b < -n$, all zeros of F are real and negative.

Proof: We use the identity (cf. [1], p.559, (15.3.4))

$$F(-n, b; c; z) = (1-z)^n F\left(-n, c-b; c; \frac{z}{z-1}\right) \tag{3.8}$$

to show that (i) ⇒ (v) and (ii) ⇒ (iv) so that it will suffice to prove (i), (ii) and (iii) above.

(i) ⇒ (v): If $b < -n$ then $c - b > c + n$ and by (i), all zeros of $F(-n, c-b; c; w)$ are real and lie in the interval $(0,1)$. Since $w = z/(z-1)$ maps $(-\infty, 0)$ to $(0,1)$, (v) follows from (3.8).

(ii) ⇒ (iv): If $-j < b < -j+1$, $j = 1,2,\ldots,n$, then $c+j-1 < c-b < c+j$, $j = 1,2,\ldots,n$. By (ii), since $w = z/(z-1)$ maps $(-\infty,0)$ to $(0,1)$ and $(1,\infty)$ to $(1,\infty)$, (iv) follows again from (3.8). To prove (i), (ii) and (iii), we note that in each part, $b > 0$ (and of course $c > 0$ by assumption). Then

$$\operatorname{sign}\binom{-b}{n} = (-1)^n, \quad \operatorname{sign}\binom{-c}{n} = (-1)^n. \tag{3.9}$$

(i) Suppose $b > c + n$. Then $b - c > n$ and

$$\operatorname{sign}\binom{b-c}{n} > 0 \text{ for all } n. \tag{3.10}$$

Considering (3.5), (3.6) and (3.7) with (3.9) and (3.10), we observe that

$$N_1 = 2\left[(X+1)/2\right], \quad N_3 = 2\left[(Z+1)/2\right],$$

$$N_2 = \begin{cases} 2\left[(Y+1)/2\right] & \text{for } n \text{ even} \\ 2\left[Y/2\right]+1 & \text{for } n \text{ odd} \end{cases}.$$

Assume now that $c > 1$. Then for $b > c+n$, we have from (3.2), (3.3) and (3.4) that $X = 0$, $Y = n$, $Z = 0$. Substituting these values into N_1, N_2 and N_3 yields the result. A similar calculation shows that the same result is obtained when $0 < c < 1$.

(ii) For $c+j-1<b<c+j$, $j=1,2,\dots,n$, we find that sign $\binom{b-c}{n}=(-1)^{n-j}$. Then from (3.5), (3.6), (3.7) we see that

$$N_1 = \begin{cases} 2\left[(X+1)/2\right] & \text{for } (n-j) \text{ even} \\ 2\left[X/2\right]+1 & \text{for } (n-j) \text{ odd} \end{cases},$$

$$N_2 = \begin{cases} 2\left[(Y+1)/2\right] & \text{for } j \text{ even} \\ 2\left[Y/2\right]+1 & \text{for } j \text{ odd} \end{cases},$$

$$N_3 = 2\left[(Z+1)/2\right].$$

It follows from (3.2), (3.3) and (3.4) by an easy calculation that $X=0$, $Y=j$, $Z=0$ and we deduce that $N_1 = \begin{cases} 0 & \text{if } (n-j) \text{ is even} \\ 1 & \text{if } (n-j) \text{ is odd} \end{cases}$, $N_2=j$ and $N_3=0$ which proves (ii).

(iii) For $0<b<c$, sign $\binom{b-c}{n}=(-1)^n$. Then $N_1 = \begin{cases} 2\left[(X+1)/2\right] & \text{if } n \text{ is even} \\ 2\left[X/2\right]+1 & \text{if } n \text{ is odd} \end{cases}$, $N_2=2\left[(Y+1)/2\right]$, $N_3=2\left[(Z+1)/2\right]$. Also, we find $X=0$, $Y=0$ and $Z=0$ which completes the proof of (iii) and hence the theorem. □

For $c<0$, the range of values of b and c that have to be considered can be reduced if we use the identities (2.1) and (2.2). Since the real zeros of $F(-n,b;c;z)$ are now known for all $c>0$ and $b\in\mathbb{R}$ from Theorem 3.2, it follows from (2.1) that we need only consider $c-b>1-n$. Similarly, from (2.2) and Theorem 3.2, we can assume $b>1-n$. We split the result for $c<0$ into the cases where $b>0$ and $1-n<b<0$.

Theorem 3.3. *Let $F=F(-n,b;c;z)$. Suppose that $c<0$, $b>0$, $c-b>1-n$. Then*

(i) $1-n<c-b<0$ and $0<b<n-1$ and $1-n<c<0$.

(ii) If $-k<c<-k+1$, $k=1,\dots,n-1$ and

$$-j<c-b<-j+1, \quad j=1,\dots,n-1,$$

then $F(-n,b;c;z)$ has $(j-k)\geq 0$ real zeros in $(0,1)$. For the remaining $(n-j+k)$ zeros of F

(a) $(n-j+k)$ are non-real if $(n-j)$ and k are even

(b) $(n-j+k-1)$ are non-real and one real zero lies in $(1,\infty)$ if $(n-j)$ is odd and k is even

(c) $(n-j+k-1)$ are non-real if $(n-j)$ is even, k odd and one zero is real and negative

(d) $(n-j+k-2)$ are non-real if $(n-j)$ is odd and k is odd with one real negative zero and one real zero in $(1,\infty)$.

Proof: (i) This follows immediately from $c<0$, $b>0$, $c-b>1-n$.

(ii) For $c < 0$, $b > 0$, $c - b > 1 - n$, we have

$$|1-c| = 1-c, \quad |b+n| = b+n, \quad |b-c-n| = c-b+n$$

and it follows from (3.2), (3.3) and (3.4) that

$$X = E(1-c-n), \quad Y = E(b), \quad Z = E(c-b).$$

Since $1-c-n < 0$ and $c - b < 0$, $X = Z = 0$. Now $\operatorname{sign}\binom{-b}{n} = (-1)^n$ and for $k = 1, \ldots, n-1$, $-k < c < -k+1 \Rightarrow \operatorname{sign}\binom{-c}{n} = (-1)^{n-k}$, while for $-j < c - b < -j+1$, $j = 1, \ldots, n-1$, $\operatorname{sign}\binom{b-c}{n} = (-1)^{n-j}$. Therefore, from (3.5), (3.6) and (3.7),

$$N_1 = \begin{cases} 0 & \text{if } (n-j) \text{ even} \\ 1 & \text{if } (n-j) \text{ odd} \end{cases} \tag{3.11}$$

$$N_2 = \begin{cases} 2\,[(Y+1)/2] & \text{if } (j-k) \text{ is even} \\ 2\,[Y/2]+1 & \text{if } (j-k) \text{ is odd} \end{cases}, \quad Y = E(b) \tag{3.12}$$

$$N_3 = \begin{cases} 0 & \text{if } k \text{ even} \\ 1 & \text{if } k \text{ odd} \end{cases}. \tag{3.13}$$

Now for $j > b - c > j-1$ and $-k < c < -k+1$, $b \in (j-k-1, j-k+1)$, $j - k = 1, 2, \ldots, n-2$. If $b \in (j-k-1, j-k)$, $Y = E(b) = j-k-1$, whereas if $b \in (j-k, j-k+1)$, $Y = E(b) = j-k$. Considering the cases $(j-k)$ even and $(j-k)$ odd, it is straight-forward to check that for all j, $k \in \mathbb{N}$ with $j - k = 0, 1, \ldots, n-2$, we have

$$N_2 = j - k. \tag{3.14}$$

Equations (3.11), (3.12), (3.13) and (3.14) complete the proof of (ii). □

By virtue of Theorem 3.3 and the identities (2.1), (2.2) and (3.8), it is easy to see that we only have one possibility left that has not been analysed, namely,

$$1-n < c-b < 0, \quad 1-n < b < 0, \quad 1-n < c < 0. \tag{3.15}$$

Theorem 3.4. *Let $F = F(-n,b;\ c;\ z)$ where b and c satisfy condition (3.15). If $-j < b < -j+1$, $j = 1, \ldots, n-1$; $-k < c < -k+1$, $k = 1, \ldots, n-1$ and $-\ell < c-b < -\ell+1$, $\ell = 1, \ldots, n-1$, then F has no real zeros if $n+j+\ell$, $k+\ell$, $j+k$ are even, one real zero in $(1,\infty)$ if $n+j+\ell$ is odd, one real zero in $(0,1)$ if $k+\ell$ is odd and one real negative zero if $j+k$ is odd.*

Proof: Under the restrictions (3.15), we have

$$|1-c| = 1-c, \quad |b+n| = b+n, \quad |b-c-n| = c-b+n.$$

Then from (3.2), (3.3) and (3.4),

$$X = E(1-c-n), \quad Y = E(b), \quad Z = E(c-b),$$

and it follows from (3.15) that $X = Y = Z = 0$. Also, sign $\binom{-b}{n} = (-1)^{n-j}$, sign $\binom{-c}{n} = (-1)^{n-k}$ and sign $\binom{b-c}{n} = (-1)^{n-\ell}$. The stated result then follows immediately from (3.5), (3.6) and (3.7). □

Remark 3.1 *We have not considered the asymptotic zero distribution as $n \to \infty$ of $F(-n, b; c; z)$. There are recent interesting results in this regard using different approaches, namely complex analysis techniques [10], matrix theoretic tools [11], asymptotic analysis of the Euler integral representation [3] and analysis of coefficients [8].*

Bibliography

1. M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, (Dover, New York, 1965).
2. Bateman Manuscript Project, *Higher Transcendental Functions, Volume I*, (A. Erdélyi, editor; McGraw–Hill, New York, 1953).
3. K. Driver and P. Duren, "Asymptotic zero distribution of hypergeometric polynomials",*Numerical Algorithms*, 21 (1999), 147–156.
4. K. Driver and P. Duren, "Zeros of the hypergeometric polynomials $F(-n, b; 2b; z)$", *Indag. Math.*, 11 (1) (2000), 43–51.
5. K. Driver and P. Duren, "Trajectories of the zeros of hypergeometric polynomials $F(-n, b; 2b; z)$ for $b < -\frac{1}{2}$", *Constr. Approx.*, 17 (2001), 169–179.
6. K. Driver and P. Duren, "Zeros of ultraspherical polynomials and the Hilbert-Klein formulas", *J. Comput. and Appl. Math.*, 135 (2001), 293–301.
7. K. Driver and M. Möller, "Quadratic and cubic transformations and the zeros of hypergeometric polynomials", *J. Comput. and Appl. Math.*, to appear.
8. K. Driver and M. Möller, "Zeros of the hypergeometric polynomials $F(-n, b; -2n; z)$", *J. Approx. Th.*, 110 (2001), 74–87.
9. F. Klein, "Über die Nullstellen der hypergeometrischen Reihe", *Mathematische Annalen*, 37 (1890), 573–590.
10. A.B.J. Kuijlaars and W. van Assche, "The asymptotic zero distribution of orthogonal polynomials with varying weights", *J. Approx. Th.*, 99 (1999), 167-197.
11. A.B.J. Kuijlaars and S. Serra Capizzano, "Asymptotic zero distribution of orthogonal polynomials with discontinuously varying recurrence coefficients", *J. Approx. Th.*, to appear.
12. A.P. Prudnikov, Yu. A. Brychkov and O.I. Marichev, *Integrals and Series, Volume 3*, (Moscow, "Nauka", 1986 (in Russian); English translation, Gordon & Breach, New York, 1988); Errata in *Math. Comp.*, 65 (1996), 1380–1384.
13. G. Szegő, *Orthogonal Polynomials*, (American Mathematical Society, New York, 1959).
14. N. Temme, *Special Functions: An introduction to the classical functions of mathematical physics*, (Wiley, New York, 1996).